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Pointwise Approximation for Linear Combinations of Bernstein Operators[1]

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For linear combinations of Bernstein operators $B_{n,r}(f, x)$, we give an equivalent theorem with $\omega_{\phi^\lambda}^{2r}(f, t)$, where $\omega_{\phi^\lambda}^{2r}(f, t)$ is the Ditzian–Totik modulus of smoothness $(1-1/r \leqslant \lambda \leqslant 1)$. It is the generalization of corresponding results by Z. Ditzian and V. Totik (1987, "Moduli of Smoothness", Springer-Verlag, Berlin/New York).

Key Words: moduli of smoothness; linear combinations; Bernstein operators.

1. INTRODUCTION

The Bernstein operator is defined by

$$B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \qquad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}. \tag{1.1}$$

The combinations of Bernstein operators introduced in [1] (see also [2] and [4]) are given by

$$B_{n,r}(f, x) = \sum_{i=0}^{r-1} C_i(n)\, B_{n_i}(f, x), \tag{1.2}$$

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where C_i and n_i satisfy

$$\begin{aligned}
&\text{(a)}\quad n=n_0<n_1<\cdots<n_{r-1}\leqslant Kn;\\
&\text{(b)}\quad \sum_{i=0}^{r-1}|C_i(n)|\leqslant C;\\
&\text{(c)}\quad \sum_{i=0}^{r-1}C_i(n)=1;\\
&\text{(d)}\quad \sum_{i=0}^{r-1}C_i(n)\,n_i^{-\rho}=0, \qquad \rho=1,2,...,r-1.
\end{aligned}\tag{1.3}$$

We recall [2, p. 10] that

$$\omega_{\phi^\lambda}^r(f,t)=\sup_{0<h\leqslant t}\ \sup_{x\pm(r/2)\,h\phi^\lambda(x)\in[0,1]}|\Delta_{h\phi^\lambda(x)}^r f(x)| \tag{1.4}$$

defined is equivalent to the K-functional [2, p. 10]

$$K_{\phi^\lambda,r}(f,t^r)=\inf_{g^{(r-1)}\in A.C._{loc}}(\|f-g\|_{C[0,1]}+t^r\,\|\phi^{r\lambda}g^{(r)}\|_{C[0,1]}). \tag{1.5}$$

That is there exists a constant C such that

$$C^{-1}K_{\phi^\lambda,r}(f,t^r)\leqslant\omega_{\phi^\lambda}^r(f,t)\leqslant CK_{\phi^\lambda,r}(f,t^r), \tag{1.6}$$

which we denote (as usual) by $\omega_{\phi^\lambda}^r(f,t)\sim K_{\phi^\lambda,r}(f,t^r)$.

In [4] we got

THEOREM A. *For* $f\in C[0,1]$, $0<\alpha<r$, $0\leqslant\lambda\leqslant1$, $\phi^2(x)=x(1-x)$, $\delta_n(x)=\phi(x)+n^{-1/2}$, *we have*

$$B_{n,r}(f,x)-f(x)=O((n^{-1/2}\delta_n^{1-\lambda}(x))^\alpha)\Leftrightarrow\omega_{\phi^\lambda}^r(f,t)=O(t^\alpha). \tag{1.7}$$

For this result, Ditzian pointed out that (see MR 99a 41028) one should note that for $\lambda=1$ the known results are substantially better, comparing $B_{n,r}(f,x)-f(x)$ with $\omega_\phi^{2r}(f,t)$ rather than with $\omega_\phi^r(f,t)$ (see [2, Chap. 9]), but this difference is inherent in the problem. For $\lambda=0$ replacing $\omega_{\phi^\lambda}^r(f,t)$ with $\omega_{\phi^\lambda}^{2r}(f,t)$ in (1.7) is impossible (see [1]). Naturally we ask for which λ we can replace $\omega_{\phi^\lambda}^r(f,t)$ with $\omega_{\phi^\lambda}^{2r}(f,t)$, for which λ we can not? The answer is given in our main result below.

THEOREM 1. *For* $f\in C[0,1]$, $r\in N$, $0<\alpha<2r$, $1-\frac{1}{r}\leqslant\lambda\leqslant1$, *we have*

$$B_{n,r}(f,x)-f(x)=O((n^{-1/2}\phi^{1-\lambda}(x))^\alpha)\Leftrightarrow\omega_{\phi^\lambda}^{2r}(f,t)=O(t^\alpha). \tag{1.8}$$

For $0\leqslant\lambda<1-1/r$, (1.8) *is not true.*

Remark 1. We also improve Theorem A by replacing $\delta_n(x)$ with $\phi(x)$.

Throughout this paper C denotes a constant independent of n and x. It is not necessarily the same at each occurrence.

2. DIRECT RESULTS WHEN $1-\frac{1}{r}<\lambda\leqslant 1$

In this section we will give direct results when $1-\frac{1}{r}<\lambda\leqslant 1$. And in the next section we will prove the case of $\lambda=1-1/r$.

LEMMA 2.1. *For $f(x)\in C[0,1]$, $r\geqslant 2$, $f^{(2r-1)}(x)\in A.C.loc$, when $1-1/r<\lambda\leqslant 1$, $m=1,2,...,r-1$ or $m=1,2,...,r-2$, $1-1/r\leqslant\lambda\leqslant 1$, we have*

$$\|\phi^{2r\lambda-2m}f^{(2r-m)}\|\leqslant C(\|f\|+\|\phi^{2r\lambda}f^{(2r)}\|), \tag{2.1}$$

where the norm $\|\cdot\|:=\|\cdot\|_{L_\infty}$.

Proof. First we observe that (see [2, p. 136])

$$\begin{aligned}|f^{(2r-m)}(\tfrac{1}{2})|&\leqslant C(\|f\|_{[1/4,\,3/4]}+\|f^{(2r)}\|_{[1/4,\,3/4]})\\&\leqslant C(\|f\|+\|\phi^{2r\lambda}f^{(2r)}\|).\end{aligned} \tag{2.2}$$

For $1-1/r<\lambda\leqslant 1$, $m=1,2,...,r-1$ or $1-1/r\leqslant\lambda\leqslant 1$, $m=1,2,...,r-2$, when x is near to $0(x\leqslant 1/2)$, we have

$$\begin{aligned}&\left|f^{(2r-m)}(x)-f^{(2r-m)}\left(\frac{1}{2}\right)\right|\\&\quad\leqslant\int_x^{1/2}|f^{(2r-m+1)}(u)|\,du\\&\quad\leqslant\|u^{r\lambda-m+1}f^{(2r-m+1)}(u)\|_{[0,\,1/2]}\cdot\int_x^{1/2}\frac{du}{u^{r\lambda-m+1}}\\&\quad\leqslant C\,\|x^{r\lambda-m+1}f^{(2r-m+1)}(x)\|_{[0,\,1/2]}\,x^{-(r\lambda-m)}\end{aligned}$$

which implies

$$\begin{aligned}&\|x^{r\lambda-m}f^{(2r-m)}(x)\|_{[0,\,1/2]}\\&\quad\leqslant C(\|f\|+\|\phi^{2r\lambda}f^{(2r)}\|+\|x^{r\lambda-m+1}f^{(2r-m+1)}(x)\|_{[0,\,1/2]}).\end{aligned}$$

When x is near to 1 $(1/2\leqslant x\leqslant 1)$, we can use similar treatment and obtain

$$\begin{aligned}&\|\phi^{2r\lambda-2m}(x)\,f^{(2r-m)}(x)\|\\&\quad\leqslant C(\|f\|+\|\phi^{2r\lambda}f^{(2r)}\|+\|\phi^{2r\lambda-2m+2}(x)\,f^{(2r-m+1)}(x)\|).\end{aligned} \tag{2.3}$$

For $m=1$ the inequality (2.1) is valid by the inequality (2.3). From these, the inequality (2.1) follows by induction.

LEMMA 2.2. *For $f(x)\in C[0,1]$, $f^{(2r-1)}(x)\in A.C.loc$, when $r\geqslant 2$, $1-1/r<\lambda\leqslant 1$, or $r=1$, $0\leqslant\lambda\leqslant 1$ we have*

$$|B_{n,r}(f,x)-f(x)|\leqslant C\frac{\phi^{2r(1-\lambda)}(x)}{n^r}(\|f\|+\|\phi^{2r\lambda}f^{(2r)}\|). \tag{2.4}$$

Proof. When $r\geqslant 2$, $1-1/r<\lambda\leqslant 1$, we discuss the inequality (2.4) by two cases.

Case 1. $x\in E_n=[1/n,1-1/n]$.

Using the Taylor expansion and [2, p. 134 (9.5.5)]

$$B_{n,r}((t-x)^j,x)=0, \qquad j=1,2,...,r \tag{2.5}$$

we can write that

$$\begin{aligned} B_{n,r}(f,x)-f(x)&=\sum_{j=1}^{r-1}\frac{1}{(2r-j)!}B_{n,r}((t-x)^{2r-j},x)\,f^{(2r-j)}(x)\\ &\quad+\frac{1}{(2r-1)!}B_{n,r}\left(\int_x^t(t-u)^{2r-1}f^{(2r)}(u)\,du,x\right)\\ &\equiv I_1+I_2. \end{aligned} \tag{2.6}$$

We estimate I_1 first. By [2] (see p. 134 (9.5.3)) and the inequality (2.1), for $x\in E_n$ one has

$$\begin{aligned} &|B_{n,r}((t-x)^{2r-j},x)\,f^{(2r-j)}(x)|\\ &\quad\leqslant C\frac{\phi^{2r(1-\lambda)}(x)}{n^r}|\phi^{2r\lambda-2j}(x)\,f^{(2r-j)}(x)|\\ &\quad\leqslant C\frac{\phi^{2r(1-\lambda)}(x)}{n^r}(\|f\|+\|\phi^{2r\lambda}f^{(2r)}\|). \end{aligned}$$

Hence

$$|I_1|\leqslant C\frac{\phi^{2r(1-\lambda)}(x)}{n^r}(\|f\|+\|\phi^{2r\lambda}f^{(2r)}\|). \tag{2.7}$$

Now we estimate I_2. From $B_n((t-x)^{2r},x)\leqslant Cn^{-r}\phi^{2r}(x)$, $(x\in E_n)$ and

$$\frac{|t-u|^{2r-1}}{\phi^{2r\lambda}(u)}\leqslant\frac{|t-x|^{2r-1}}{\phi^{2r\lambda}(x)} \tag{2.8}$$

for u is between x and t (see [2, p. 128], [2, p. 141]), we have

$$\left| B_{n,r}\left(\int_x^t (t-u)^{2r-1} f^{(2r)}(u)\,du, x \right) \right|$$
$$\leqslant \sum_{i=0}^{r-1} |C_i(n)|\, \|\phi^{2r\lambda} f^{(2r)}\|\, B_{n_i}\left(\frac{|t-x|^{2r}}{\phi^{2r\lambda}(x)}, x \right)$$
$$\leqslant C \frac{\phi^{2r(1-\lambda)}(x)}{n^r} \|\phi^{2r\lambda} f^{(2r)}\|. \tag{2.9}$$

So

$$|I_2| \leqslant C \frac{\phi^{2r(1-\lambda)}(x)}{n^r} \|\phi^{2r\lambda} f^{(2r)}\|. \tag{2.10}$$

Case 2. $x \in E_n^c = [0, 1/n) \cup (1-1/n, 1]$.
First we write

$$f(t) = f(x) + (t-x) f'(x) + \cdots + \frac{1}{r!} f^{(r)}(x)(t-x)^r$$
$$+ \frac{1}{r!} \int_x^t (t-u)^r f^{(r+1)}(u)\,du.$$

From the inequality (2.8) and Lemma 2.1 we get for $1 - 1/r < \lambda \leqslant 1$

$$|B_{n,r}(f, x) - f(x)|$$
$$\leqslant \phi^{2(r-1)-2r\lambda}(x)\, \|\phi^{2r\lambda-2(r-1)} f^{(r+1)}\| \sum_{i=0}^{r-1} |C_i(n)|\, B_{n_i}(|t-x|^{r+1}, x)$$
$$\leqslant C\phi^{2r(1-\lambda)-2}(x)(\|f\| + \|\phi^{2r\lambda} f^{(2r)}\|) \sum_{i=0}^{r-1} |C_i(n)|\, B_{n_i}(|t-x|^{r+1}, x).$$

By [2, (9.5.10)], for $x \in E_n^c$, we have $B_n(|t-x|^{2r}, x) \leqslant C \frac{\phi^2(x)}{n^{2r-1}}$, therefore

$$B_n(|t-x|^{r+1}, x) \leqslant (B_n((t-x)^{2r}, x)\, B_n((t-x)^2, x))^{1/2} \leqslant C \frac{\phi^2(x)}{n^r}.$$

Hence for $x \in E_n^c$ we have

$$|B_{n,r}(f, x) - f(x)| \leqslant C \frac{\phi^{2r(1-\lambda)}(x)}{n^r} (\|f\| + \|\phi^{2r\lambda} f^{(2r)}\|). \tag{2.11}$$

From (2.6), (2.7), (2.10) and (2.11), (2.4) follows.

When $r=1$, $0\leqslant\lambda\leqslant 1$,

$$|B_n(f,x)-f(x)|=\left|B_n\left(\int_x^t (t-u)\,f''(u)\,du,\,x\right)\right|$$

$$\leqslant \|\varphi^{2\lambda}f''\|\,\varphi^{-2\lambda}(x)\,B_n((t-x)^2,x)=\frac{\varphi^{2(1-\lambda)}(x)}{n}\,\|\varphi^{2\lambda}f''\|.$$

Lemma 2.2 is proved.

THEOREM 2. *For* $f\in C[0,1]$, $1-1/r<\lambda\leqslant 1 (r\geqslant 2)$, *or* $0\leqslant\lambda\leqslant 1$ $(r=1)$ *then*

$$|B_{n,r}(f,x)-f(x)|\leqslant C\left(\frac{\phi^{2r(1-\lambda)}(x)}{n^r}\,\|f\|+\omega^{2r}_{\phi^\lambda}\left(f,\frac{\phi^{1-\lambda}(x)}{\sqrt{n}}\right)\right). \tag{2.12}$$

Proof. By (1.6), we may choose $g_n\equiv g_{n,x,\lambda}$ for a fixed x and λ such that

$$\|f-g_n\|+(n^{-1/2}\phi^{1-\lambda}(x))^{2r}\,\|\phi^{2r\lambda}g_n^{(2r)}\|\leqslant C\omega^{2r}_{\phi^\lambda}(f,n^{-1/2}\phi^{1-\lambda}(x)). \tag{2.13}$$

From the definition of the $B_{n,r}$ and Lemma 2.2, we have

$$\begin{aligned}
&|B_{n,r}(f,x)-f(x)|\\
&\quad\leqslant C\,\|f-g_n\|+|B_{n,r}(g_n,x)-g_n(x)|\\
&\quad\leqslant C\left(\|f-g_n\|+\frac{\phi^{2r(1-\lambda)}(x)}{n^r}\,\|g_n\|+\omega^{2r}_{\phi^\lambda}\left(f,\frac{\phi^{1-\lambda}(x)}{\sqrt{n}}\right)\right)\\
&\quad\leqslant C\left(\frac{\phi^{2r(1-\lambda)}(x)}{n^r}\,\|f\|+\omega^{2r}_{\phi^\lambda}\left(f,\frac{\phi^{1-\lambda}(x)}{\sqrt{n}}\right)\right).
\end{aligned}$$

Remark 2. [2, (9.3.1)] is the special case of (2.12) for $\lambda=1$.

Remark 3. For $0\leqslant\lambda<1-1/r$, (2.12) is not true.
For $f(x)=x^{r+1}$, let $x=1/n$. Then

$$\omega^{2r}_{\phi^\lambda}\left(f,\frac{\phi^{1-\lambda}(x)}{\sqrt{n}}\right)+\frac{\phi^{2r(1-\lambda)}(x)}{n^r}\,\|f\|\sim\left(\frac{1}{n}\right)^{r(2-\lambda)}.$$

In the case of $r=2j$ $(j=1,2,...)$, using [2, (9.5.11)], we have

$$\begin{aligned}
B_{n,r}(f,x)-f(x)&=B_{n,r}((t-x)^{r+1},x)\\
&=\sum_{m=0}^{j-1}\frac{\phi^{2j-2m}(x)}{n^{j+m+1}}\,P_m(x)\sim\left(\frac{1}{n}\right)^{2j+1}.
\end{aligned}$$

In the case of $r=2j-1$, using [2, (9.5.10)], similarly we have

$$B_{n,r}(f,x)-f(x)\sim\left(\frac{1}{n}\right)^{2j}.$$

So for $0\leqslant\lambda<1-1/r$, (2.12) is not valid and in (1.8) the relation "$\Leftarrow$" is not true.

3. DIRECT THEOREM WHEN $\lambda=1-1/r$

LEMMA 3.1. *Let $0<\alpha<2r$, If $\omega_{\phi^\lambda}^{2r}(f,t)=O(t^\alpha)$, $\lambda=1-1/r$, then*

$$\omega^{r+1}(f,t)=O(t^{\alpha(1-\lambda/2)}), \tag{3.1}$$

where $\omega^{r+1}(f,t)$ is the classical modulus of smoothness.

Proof. By the following relation (see (3.1.5) of [2])

$$\omega^r(f,t^{1/(1-\lambda/2)})\leqslant M\omega_{\phi^\lambda}^r(f,t),$$

we can deduce

$$\omega^{2r}(f,t)=\omega^{2r}(f,(t^{1-\lambda/2})^{1/(1-\lambda/2)})\leqslant M\omega_{\phi^\lambda}^{2r}(f,t^{1-\lambda/2})\leqslant Ct^{\alpha(1-\lambda/2)}.$$

And because of $0<\alpha<2r$, $0<\alpha(1-\lambda/2)<r+1$, then using above inequality and the following relation (see (4.3.1) of [2])

$$\omega^r(f,t)\leqslant Ct^r\left\{\int_t^c\frac{\omega^{r+1}(f,u)}{u^{r+1}}\,du+\|f\|\right\},$$

where c is a positive constant, we can obtain

$$\omega^{r+1}(f,t)\leqslant Ct^{\alpha(1-\lambda/2)}.$$

LEMMA 3.2. *For $f(x)\in C[0,1]$, $f^{(2r-1)}(x)\in A.C.loc$, $r\geqslant 2$, when $x\in E_n$, $\lambda=1-1/r$, we have*

$$\begin{aligned}|B_{n,r}(f,x)-f(x)|&\leqslant C\omega^{r+1}(f,(n^{-r}\phi^{2r(1-\lambda)}(x))^{1/(r+1)})\\&\quad+Cn^{-r}\phi^{2r(1-\lambda)}(x)(\|f\|+\|\phi^{2r\lambda}f^{(2r)}\|).\end{aligned} \tag{3.2}$$

Proof. Let

$$T_{n,r+1}(f,x)=\frac{-1}{(r+1)!}(\operatorname{Sgn}R_{n,r+1}(x))\,\vec{\Delta}^{r+1}_{|R_{n,r+1}(x)|^{1/(r+1)}}f(x),$$

where $R_{n,r+1}(x)=B_{n,r}((t-x)^{r+1},x)$, $\vec{\Delta}_t^1 f(x)\equiv f(x+t)-f(x)$, $\vec{\Delta}_t^k f(x)\equiv \vec{\Delta}(\vec{\Delta}_t^{k-1} f(x))$. By simple calculation we know

$$T_{n,r+1}((t-x)^j,x)=\begin{cases}0, & j<r+1,\\ -R_{n,r+1}(x), & j=r+1,\\ c_j\,|R_{n,r+1}(x)|^{j/(r+1)}\,(\text{Sgn}\,R_{n,r+1}(x)), & j>r+1,\end{cases}$$

where c_j is a constant that depends on j but not on n and x.

On the other hand, when $x\in E_n$, we have

$$|R_{n,r+1}(x)|\leqslant Cn^{-r}\phi^2(x)=Cn^{-r}\phi^{2r(1-\lambda)}(x),$$

$$|T_{n,r+1}(f,x)|\leqslant C\omega^{r+1}(f,(n^{-r}\phi^{2r(1-\lambda)}(x))^{1/(r+1)}),$$

and

$$\begin{aligned}|T_{n,r+1}((t-x)^j,x)|&=|c_j|\cdot|R_{n,r+1}(x)|^{j/(r+1)}\leqslant C(n^{-r}\phi^2(x))^{j/(r+1)}\\ &\leqslant Cn^{-r}\phi^{2(j-r)}(x)\,n^{r(1-j/(r+1))}\phi^{2r(1-j/(r+1))}(x)\\ &\leqslant Cn^{-r}\phi^{2(j-r)}(x),\qquad (j>r+1).\end{aligned}\tag{3.3}$$

Now we define a new operator $A_n(f,x)=T_{n,r+1}(f,x)+B_{n,r}(f,x)$, then

$$A_n((t-x)^j,x)=0,\qquad j=1,2,...,r,r+1.$$

Similar to Lemma 2.2 we write that

$$\begin{aligned}|A_n(f,x)-f(x)|\leqslant&\left|\sum_{j=1}^{r-2}\frac{1}{(2r-j)!}B_{n,r}((t-x)^{2r-j},x)\,f^{(2r-j)}(x)\right|\\ &+\left|\frac{1}{(2r-1)!}B_{n,r}\left(\int_x^t(t-u)^{2r-1}f^{(2r)}(u)\,du,x\right)\right|\\ &+\left|\sum_{j=1}^{r-2}\frac{1}{(2r-j)!}T_{n,r+1}((t-x)^{2r-j},x)\,f^{(2r-j)}(x)\right|\\ &+\left|\frac{1}{(2r-1)!}T_{n,r+1}\left(\int_x^t(t-u)^{2r-1}f^{(2r)}(u)\,du,x\right)\right|\\ \equiv&\,J_1+J_2+J_3+J_4,\end{aligned}$$

(when $r=2$, $J_1=0$, $J_3=0$).

By the procedure of the proof of Lemma 2.2 we know that

$$J_1+J_2\leqslant Cn^{-r}\phi^{2r(1-\lambda)}(x)(\|f\|+\|\phi^{2r\lambda}f^{(2r)}\|).$$

Using (3.3), on a similar plan of (2.7) we can get

$$J_3 \leqslant Cn^{-r}\phi^{2r(1-\lambda)}(x)(\|f\| + \|\phi^{2r\lambda}f^{(2r)}\|).$$

Now we estimate J_4. We know

$$\begin{aligned}|T_{n,r+1}(f,x)| &= \frac{-1}{(r+1)!}\,|\vec{\Delta}^{r+1}_{|R_{n,r+1}(x)|^{1/(r+1)}}f(x)|\\ &= \frac{-1}{(r+1)!}\left|\sum_{m=0}^{r+1}(-1)^m\binom{r+1}{m}\right.\\ &\quad\left.\times f(x+(r+1-m)\,|R_{n,r+1}(x)|^{1/(r+1)})\right|,\end{aligned}$$

so

$$\begin{aligned}J_4 \leqslant \sum_{m=0}^{r+1}\binom{r+1}{m}&\int_x^{x+(r+1-m)\,|R_{n,r+1}(x)|^{1/(r+1)}}\\ &\times(x+(r+1-m)\,|R_{n,r+1}(x)|^{1/(r+1)}-u)^{2r-1}\,|f^{2r}(u)|\,du.\end{aligned}$$

Then similar to the proof (2.9), we can deduce by (2.8) and (3.3)

$$J_4 \leqslant Cn^{-r}\phi^{2r(1-\lambda)}(x)(\|f\| + \|\phi^{2r\lambda}f^{(2r)}\|).$$

Therefore

$$|A_n(f,x)-f(x)| \leqslant Cn^{-r}\phi^{2r(1-\lambda)}(x)(\|f\| + \|\phi^{2r\lambda}f^{(2r)}\|).$$

Thus we obtain

$$\begin{aligned}|B_{n,r}(f,x)-f(x)| &\leqslant |A_n(f,x)-f(x)| + |T_{n,r+1}(f,x)|\\ &\leqslant C\omega^{r+1}(f,(n^{-r}\phi^2(x))^{1/(r+1)})\\ &\quad + Cn^{-r}\phi^{2r(1-\lambda)}(x)(\|f\| + \|\phi^{2r\lambda}f^{(2r)}\|).\end{aligned}$$

Lemma 3.2 has been proved.

Similar to the proof of Theorem 2, we can obtain the following theorem.

THEOREM 3. *Let* $f \in C[0,1]$, *when* $x \in E_n$, $\lambda = 1-1/r$, *we have*

$$\begin{aligned}&|B_{n,r}(f,x)-f(x)|\\ &\quad\leqslant C\omega^{r+1}(f,(n^{-r}\phi^{2r(1-\lambda)}(x))^{1/(r+1)})\\ &\qquad + C\left(\frac{\phi^{2r(1-\lambda)}(x)}{n^r}\,\|f\| + \omega^{2r}_{\phi^\lambda}(f, n^{-1/2}\phi^{1-\lambda}(x))\right).\end{aligned} \tag{3.4}$$

To discuss the case of $x \in E_n^c$, we define for $h > 0$ the Steklov-type averages

$$f_h(x) = \left(\frac{r+1}{h}\right)^{r+1} \int_0^{h/r+1} \cdots \int_0^{h/r+1}$$

$$\times \left\{ \sum_{k=1}^{r+1} \binom{r+1}{k} (-1)^{k+1} f(x + k(u_1 + \cdots + u_{r+1})) \right\} du_1 \cdots du_{r+1}.$$

We know $f_h(x)$ has $r+1$ continuous derivatives. And when $[x, x+(r+1)h] \subset [0, 1]$, by calculation we have

$$|f(x) - f_h(x)| \leqslant C\omega^{r+1}(f, h),$$

$$|f_h^{(r+1)}(x)| \leqslant Ch^{-(r+1)}\omega^{r+1}(f, h).$$

Then we choose a function $\psi \in C^\infty$ such that $\psi(x) = 1$ on $[0, 1/3]$, $\psi(x) = 0$ on $[2/3, 1]$ and $\psi(x)$ is decreasing. Let $F_h(x) = f_h(x)\,\psi(x) + f_{-h}(1 - \psi(x))$, where f_{-h} is the same as f_h but using $-h$ instead of h. Using the standard technique of [3, p. 106], we can deduce for $x \in [0, 1]$

$$|f(x) - F_h(x)| \leqslant C\omega^{r+1}(f, h), \tag{3.5}$$

$$|F_h^{(r+1)}(x)| \leqslant Ch^{-(r+1)}\omega^{r+1}(f, h). \tag{3.6}$$

Therefore similar to the case 2 of Lemma 2.2 we also have for $x \in E_n^c$ by (3.6)

$$\begin{aligned} |B_{n,r}(F_h, x) - F_h(x)| &= \frac{1}{r!} \left| B_{n,r}\left(\int_x^t (t-u)^r F_h^{(r+1)}(u)\, du, x \right) \right| \\ &\leqslant Ch^{-(r+1)}\omega^{r+1}(f, h) \sum_{i=0}^{r-1} |C_i(n)|\, B_{n_i}(|t-x|^{r+1}, x) \\ &\leqslant Ch^{-(r+1)}\omega^{r+1}(f, h)\, n^{-r}\varphi^2(x). \end{aligned} \tag{3.7}$$

Now we give the direct theorem:

THEOREM 4. *Let* $f \in C[0, 1]$, *when* $x \in [0, 1]$, $\lambda = 1 - 1/r$, $0 < \alpha < 2r$, *if* $\omega_{\phi^\lambda}^{2r}(f, t) = O(t^\alpha)$, *then we have*

$$|B_{n,r}(f, x) - f(x)| = O((n^{-1/2}\phi^{1-\lambda}(x))^\alpha) = O(n^{-\alpha/2}\phi^{\alpha/r}(x)). \tag{3.8}$$

Proof. We will prove (3.8) by two cases.

Case 1. When $x \in E_n$, using Lemma 3.1 and Theorem 3 we obtain

$$\begin{aligned}
&|B_{n,r}(f,x)-f(x)| \\
&\quad \leqslant C\omega^{r+1}(f,(n^{-r}\phi^2(x))^{1/(r+1)})+C\omega_{\phi^\lambda}^{2r}(f,n^{-1/2}\phi^{1-\lambda}(x)) \\
&\quad \leqslant C(n^{-r}\phi^2(x))^{(\alpha/(r+1))(1-\lambda/2)}+C(n^{-1/2}\phi^{1-\lambda}(x))^\alpha \\
&\quad \leqslant C(n^{-1/2}\phi^{1-\lambda}(x))^\alpha.
\end{aligned}$$

Case 2. When $x \in E_n^c$, using (3.5), (3.7) and Lemma 3.1, and choosing $h=(n^{-r}\phi^2(x))^{1/r+1}$ we have

$$\begin{aligned}
&|B_{n,r}(f,x)-f(x)| \\
&\quad \leqslant |B_{n,r}(f-F_h,x)|+|f(x)-F_h(x)|+|B_{n,r}(F_h,x)-F_h(x)| \\
&\quad \leqslant C\omega^{r+1}(f,h)+Ch^{-(r+1)}\omega^{r+1}(f,h)\,n^{-r}\phi^2(x) \\
&\quad \leqslant Ch^{\alpha(1-\lambda/2)}+Ch^{-(r+1)}n^{-r}\phi^2(x)\,h^{\alpha(1-\lambda/2)} \\
&\quad \leqslant C(n^{-1/2}\phi^{1-\lambda}(x))^\alpha.
\end{aligned}$$

Theorem 4 has been proved.

Remark 4. In fact by this method we can also deal with the case of $1-1/r<\lambda\leqslant 1$ in Theorem 1. But we cannot obtain Theorem 2 (a better direct theorem).

4. INVERSE RESULTS

THEOREM 5. *For $f \in C[0,1]$, $r \in N$, $0<\alpha<2r$, $0 \leqslant \lambda \leqslant 1$, if*

$$|B_{n,r}(f,x)-f(x)| \leqslant C((n^{-1/2}\phi^{1-\lambda}(x))^\alpha), \tag{4.1}$$

then

$$\omega_{\phi^\lambda}^{2r}(f,t)=O(t^\alpha). \tag{4.2}$$

Proof. From the procedure of proof in [4, Theorem 2], we can deduce that for $0<\alpha<2r$

$$|B_{n,r}(f,x)-f(x)|=O((n^{-1/2}\delta_n^{1-\lambda}(x))^\alpha) \Rightarrow \omega_{\phi^\lambda}^{2r}(f,t)=O(t^\alpha). \tag{4.3}$$

Since (4.1) implies the left of (4.3), (4.2) follows.

Remark 5. Obviously, by Theorems 2, 4 and 5, noting Remark 3 we know that Theorem 1 is true.

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